Some computations in the cyclic permutations of completely rational nets

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Abstract

In this paper we calculate certain chiral quantities from the cyclic permutation orbifold of a general completely rational net. We determine the fusion of a fundamental soliton, and by suitably modified arguments of A. Coste , T. Gannon and especially P. Bantay to our setting we are able to prove a number of arithmetic properties including congruence subgroup properties for S, T matrices of a completely rational net defined by K.-H. Rehren . 2000MSC:81R15, 17B69.

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1 Introduction

Let $\mathcal{A}$ be a completely rational net (cf. Definition 2.6). Then $\mathcal{A} \otimes \mathcal{A} \otimes ... \otimes \mathcal{A}$ (n tensors) admits an action of $\mathbb{Z}_n$ by cyclic permutations. The corresponding orbifold net is referred to as the (n-th order) cyclic permutation orbifold of $\mathcal{A}$. This construction has been used both in mathematics and physics literature (for a partial list, see [1], [2], [3], [5] and references therein). In [23], this construction was used for $n = 2$ to show that strong additivity is automatic in a conformal net with finite μ index. In [18], the construction was used to demonstrate applications of general orbifold theories among other things. The starting motivation of this paper is to improve a result (Prop. 9.4 in [18]) on fusion of a fundamental soliton. The second motivation came from two papers: one by A. Coste and T. Gannon (cf. [7]) where under certain conditions they showed that the S, T matrices verified congruence subgroup properties, and one by P. Bantay (cf. [1])where he showed that congruence subgroup properties hold if a number a heuristic arguments including what he called "Orbifold Covariance Principle" hold. This "Orbifold Covariance Principle" of P. Bantay is highly nontrivial even in concrete examples, and at present the only conceptual framework in which this principle is a theorem is in the framework of local conformal net (cf. Section 2.1), where the principle follows from Theorem 2.7. In the language of local conformal nets the S, T matrices were defined by K.-H. Rehren (cf. [26]) by using local data of conformal nets, and in all known cases they agree with the "S,T" matrices coming from modular transformations of characters. If one is interested in the modular tensor categories, then this "S,T" matrices of Rehren are sufficient for calculations of three manifold invariants (cf. for example [30]). It is therefore an interesting question to see if one can adapt the methods of A. Coste and T. Gannon and P. Bantay to Rehren's "S,T" matrices.

In this paper we will show that Prop. 9.4 of [18] holds in general (cf. Th. 3.6), and that a suitable modification of the arguments of A. Coste and T. Gannon and P. Bantay is possible in the conformal net setting, and congruence subgroup properties hold for Rehern's "S,T" matrices (cf. Th. 4.9). Our key observation is the squares of nets in §3. By using (3) of Lemma 2.10 which relates the chiral data of a net and subnet for suitably chosen squares, we are able to obtain strong constraints on certain matrices (cf. Th. 3.12). These squares are in fact commuting squares first considered in the setting of II_1 factors by S. Popa in [24], and they already played an important role in the setting of nets in [33]. However the "commuting" property of these squares will not play an explicit role in this paper. Th. 3.12 allows us to apply the methods of P. Bantay in [1] to obtain arithmetic properties of Rehren's "S,T" matrices in Th. 4.5 and Th. 4.9.

We note that Th. 3.12 implies series of arithmetic properties of "S,T" matrices, and even the first one as observed in [18] seems to be nontrivial for concrete examples like the nets coming from $SU(n)$ at level k.

The rest of this paper is organized as follows: In §2, after recalling basic definitions of completely rational nets, Rehren's S, T matrices, orbifolds and Galois actions, we stated a few general results from §9 of [18] which will be used in §3 and §4. In §3

we improve Prop. 9.4 of [18] in Th. 3.6, and we present the proof of Th. 3.12 as mentioned above from a commuting square. In §4, by modifications incorporating phase factors the arguments of A. Coste and T. Gannon and P. Bantay, we are able to prove Th. 4.5 and Th. 4.9. We note that the arguments in §4 can be simplified if one can prove a conjecture on Page 734 of [18].

2 Conformal nets, complete rationality, and orbifolds

For the convenience of the reader we collect here some basic notions that appear in this paper. This is only a guideline and the reader should look at the references for a more complete treatment.

2.1 Conformal nets on S^1

By an interval of the circle we mean an open connected non-empty subset I of S^1 such that the interior of its complement I' is not empty. We denote by $\mathcal{I}$ the family of all intervals of S^1.

A *net* $\mathcal{A}$ of von Neumann algebras on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

from $\mathcal{I}$ to von Neumann algebras on a fixed Hilbert space $\mathcal{H}$ that satisfies:

A. *Isotony.* If $I_1 \subset I_2$ belong to $\mathcal{I}$, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2).$$

The net $\mathcal{A}$ is called *local* if it satisfies:

B. *Locality.* If $I_1, I_2 \in \mathcal{I}$ and $I_1 \cap I_2 = \varnothing$ then

$$[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\},$$

where brackets denote the commutator.

The net $\mathcal{A}$ is called *Möbius covariant* if in addition satisfies the following properties **C,D,E,F**:

C. *Möbius covariance.* There exists a strongly continuous unitary representation U of the Möbius group $\mathsf{Möb}$ (isomorphic to $PSU(1,1)$) on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* \ = \ \mathcal{A}(gI), \quad g \in \mathsf{Möb}, \ I \in \mathcal{I}.$$

If $E \subset S^1$ is any region, we shall put $\mathcal{A}(E) \equiv \bigvee_{E\supset I\in\mathcal{I}} \mathcal{A}(I)$ with $\mathcal{A}(E) = \mathbb{C}$ if E has empty interior (the symbol $\vee$ denotes the von Neumann algebra generated). Note that the definition of $\mathcal{A}(E)$ remains the same if E is an interval namely: if $\{I_n\}$ is an increasing sequence of intervals and $\cup_n I_n = I$, then the $\mathcal{A}(I_n)$'s generate $\mathcal{A}(I)$ (consider a sequence of elements $g_n \in \mathsf{M\ddot{o}b}$ converging to the identity such that $g_n I \subset I_n$).

D. *Positivity of the energy.* The generator of the one-parameter rotation subgroup of U (conformal Hamiltonian) is positive.

E. *Existence of the vacuum.* There exists a unit U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$.

By the Reeh-Schlieder theorem Ω is cyclic and separating for every fixed $\mathcal{A}(I)$. The modular objects associated with $(\mathcal{A}(I), \Omega)$ have a geometric meaning

$$\Delta_I^{it} = U(\Lambda_I(2\pi t)), \qquad J_I = U(r_I)\ .$$

Here Λ_I is a canonical one-parameter subgroup of $\mathsf{M\ddot{o}b}$ and $U(r_I)$ is a antiunitary acting geometrically on $\mathcal{A}$ as a reflection r_I on S^1.

This implies *Haag duality*:

$$\mathcal{A}(I)' = \mathcal{A}(I'), \quad I \in \mathcal{I}\ ,$$

where I' is the interior of $S^1 \smallsetminus I$.

F. *Irreducibility.* $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$. Indeed $\mathcal{A}$ is irreducible iff Ω is the unique U-invariant vector (up to scalar multiples). Also $\mathcal{A}$ is irreducible iff the local von Neumann algebras $\mathcal{A}(I)$ are factors. In this case they are III_1-factors in Connes classification of type III factors (unless $\mathcal{A}(I) = \mathbb{C}$ for all I).

By a *conformal net* (or diffeomorphism covariant net) $\mathcal{A}$ we shall mean a Möbius covariant net such that the following holds:

G. *Conformal covariance.* There exists a projective unitary representation U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of $\mathsf{M\ddot{o}b}$ such that for all $I \in \mathcal{I}$ we have

$$\begin{aligned} U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in \mathrm{Diff}(S^1), \\ U(g)xU(g)^* &= x, \quad x \in \mathcal{A}(I),\ g \in \mathrm{Diff}(I'), \end{aligned}$$

where $\mathrm{Diff}(S^1)$ denotes the group of smooth, positively oriented diffeomorphism of S^1 and $\mathrm{Diff}(I)$ the subgroup of diffeomorphisms g such that $g(z) = z$ for all $z \in I'$.

Let G be a simply connected compact Lie group. By Th. 3.2 of [12], the vacuum positive energy representation of the loop group LG (cf. [25]) at level k gives rise to an irreducible conformal net denoted by $\mathcal{A}_{G_k}$. By Th. 3.3 of [12], every irreducible positive energy representation of the loop group LG at level k gives rise to an irreducible covariant representation of $\mathcal{A}_{G_k}$.

2.2 Doplicher-Haag-Roberts superselection sectors in CQFT

The references of this section are [10, 11, 20, 21, 22, 13, 14]. The DHR theory was originally made on the 4-dimensional Minkowski spacetime, but can be generalized to our setting. There are however several important structure differences in the low dimensional case.

A (DHR) representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I} \mapsto \pi_I$ that associates to each I a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_{\tilde{I}} \restriction \mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \subset \mathcal{I} \ .$$

π is said to be Möbius (resp. diffeomorphism) covariant if there is a projective unitary representation U_π of $\mathsf{M\ddot{o}b}$ (resp. $\mathrm{Diff}^{(\infty)}(S^1)$, the infinite cover of $\mathrm{Diff}(S^1)$) on $\mathcal{H}$ such that

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$ and $g \in \mathsf{M\ddot{o}b}$ (resp. $g \in \mathrm{Diff}^{(\infty)}(S^1)$). Note that if π is irreducible and diffeomorphism covariant then U is indeed a projective unitary representation of $\mathrm{Diff}(S^1)$.

By definition the irreducible conformal net is in fact an irreducible representation of itself and we will call this representation the *vacuum representation*.

Given an interval I and a representation π of $\mathcal{A}$, there is an *endomorphism of $\mathcal{A}$ localized in I* equivalent to π; namely ρ is a representation of $\mathcal{A}$ on the vacuum Hilbert space $\mathcal{H}$, unitarily equivalent to π, such that $\rho_{I'} = \mathrm{id} \restriction \mathcal{A}(I')$.

Fix an interval I_0 and endomorphisms ρ, ρ' of $\mathcal{A}$ localized in I_0. Then the *composition* (tensor product) $\rho\rho'$ is defined by

$$(\rho\rho')_I = \rho_I \rho'_I$$

with I an interval containing I. One can indeed define $(\rho\rho')_I$ for an arbitrary interval I of S^1 (by using covariance) and get a well defined endomorphism of $\mathcal{A}$ localized in I_0. Indeed the endomorphisms of $\mathcal{A}$ localized in a given interval form a tensor C^*-category. For our needs ρ, ρ' will be always localized in a common interval I.

If π and π' are representations of $\mathcal{A}$, fix an interval I_0 and choose endomorphisms ρ, ρ' localized in I_0 with ρ equivalent to π and ρ' equivalent to π'. Then $\pi \cdot \pi'$ is defined (up to unitary equivalence) to be $\rho\rho'$. The class of a DHR representation modulo unitary equivalence is a *superselection sectors* (or simply a sector). We use the notations $\rho_1 \succ \rho_2$ or $\rho_2 \prec \rho_1$ if ρ_2 appears a summand of ρ_1.

The localized endomorphisms of $\mathcal{A}$ for a tensor C^*-category. For our needs, ρ, ρ' will be always localized in a common interval I.

We now define the statistics. Given the endomorphism $\rangle$ of $\mathcal{A}$ localized in $I \in \mathcal{I}$, choose an equivalent endomorphism $\rangle_0$ localized in an interval $I_0 \in \mathcal{I}$ with $\bar{I}_0 \cap \bar{I} = \varnothing$ and let u be a local intertwiner in $\mathrm{Hom}(\rangle, \rangle_0)$ as above, namely $u \in \mathrm{Hom}(\rho_{\tilde{I}}, \rho_{0,\tilde{I}})$ with I_0 following clockwise I inside $\tilde{I}$ which is an interval containing both I and I_0.

The *statistics operator* $\varepsilon := u^*\rho(u) = u^*\rho_{\tilde{I}}(u)$ belongs to $\mathrm{Hom}(\rho^2_{\tilde{I}}, \rho^2_{\tilde{I}})$. An elementary computation shows that it gives rise to a presentation of the Artin braid group

$$\epsilon_i\epsilon_{i+1}\epsilon_i = \epsilon_{i+1}\epsilon_i\epsilon_{i+1}, \qquad \epsilon_i\epsilon_{i'} = \epsilon_{i'}\epsilon_i \quad \text{if } |i - i'| \geq 2,$$

where $\varepsilon_i = \rho^{i-1}(\varepsilon)$. The (unitary equivalence class of the) representation of the Artin braid group thus obtained is the *statistics* of the superselection sector ρ.

It turns out the endomorphisms localized in a given interval form a *braided C^*-tensor category* with unitary braiding.

The *statistics parameter* λ_ρ can be defined in general. In particular, assume ρ to be localized in I and $\rho_I \in \mathrm{End}((\mathcal{A}(I))$ to be irreducible with a conditional expectation $E : \mathcal{A}(I) \to \rho_I(\mathcal{A}(I))$, then

$$\lambda_\rho := E(\epsilon)$$

depends only on the superselection sector of ρ.

The *statistical dimension* $d(\rho)$ and the *univalence* ω_ρ are then defined by

$$d(\rho) = |\lambda_\rho|^{-1} , \qquad \omega_\rho = \frac{\lambda_\rho}{|\lambda_\rho|} .$$

The *conformal spin-statistics theorem* shows that

$$\omega_\rho = e^{i2\pi\Delta_\rho} ,$$

where Δ_ρ is the conformal dimension (the lowest eigenvalue of the generator of the rotation subgroup) in the representation ρ. The right hand side in the above equality is called the *univalence* of ρ.

$d(\rho)^2$ will be called the index of ρ. The general index was first defined and investigated by Vaughan Jones in the case of II_1 factors in [16].

2.3 Rehren's S, T-matrices

Next we will recall some of the results of [26] and introduce notations.

Let $\{[\lambda], \lambda \in \mathcal{P}\}$ be a finite set of all equivalence classes of irreducible, covariant, finite-index representations of an irreducible local conformal net $\mathcal{A}$. We will denote the conjugate of $[\lambda]$ by $[\bar{\lambda}]$ and identity sector (corresponding to the vacuum representation) by $[1]$ if no confusion arises, and let $N^\nu_{\lambda\mu} = \langle [\lambda][\mu], [\nu]\rangle$. Here $\langle \mu, \nu\rangle$ denotes the dimension of the space of intertwiners from μ to ν (denoted by $\mathrm{Hom}(\mu, \nu)$). We will denote by $\{T_e\}$ a basis of isometries in $\mathrm{Hom}(\nu, \lambda\mu)$. The univalence of λ and the statistical dimension of (cf. §2 of [13]) will be denoted by ω_λ and $d(\lambda)$ (or d_λ)) respectively.

Let φ_λ be the unique minimal left inverse of λ, define:

$$Y_{\lambda,\mu} := d(\lambda)d(\mu)\varphi_\mu(\epsilon(\mu, \lambda)^*\epsilon(\lambda, \mu)^*), \tag{1}$$

where $\epsilon(\mu, \lambda)$ is the unitary braiding operator (cf. [13]).

We list two properties of $Y_{\lambda,\mu}$ (cf. (5.13), (5.14) of [26]) which will be used in the following:

Lemma 2.1.

$$Y_{\lambda,\mu} = Y_{\mu,\lambda} = Y^*_{\lambda,\bar{\mu}} = Y_{\bar{\lambda},\bar{\mu}}.$$
$$Y_{\lambda,\mu} = \sum_k N^\nu_{\lambda\mu} \frac{\omega_\lambda \omega_\mu}{\omega_\nu} d(\nu).$$

We note that one may take the second equation in the above lemma as the definition of $Y_{\lambda,\mu}$.

Define $a := \sum_i d^2_{\rho_i} \omega^{-1}_{\rho_i}$. If the matrix $(Y_{\mu,\nu})$ is invertible, by Proposition on P.351 of [26] a satisfies $|a|^2 = \sum_\lambda d(\lambda)^2$.

Definition 2.2. *Let $a = |a| \exp(-2\pi i \frac{c_0(\mathcal{A})}{8})$ where $c_0(\mathcal{A}) \in \mathbb{R}$ and $c_0(\mathcal{A})$ is well defined* mod $8\mathbb{Z}$. *For simplicity we will denote $c_0(\mathcal{A})$ simply as c_0 when the underlying $\mathcal{A}$ is clear.*

Define matrices

$$S := |a|^{-1} Y, T := C \mathrm{Diag}(\omega_\lambda) \tag{2}$$

where

$$C := \exp(-2\pi i \frac{c_0}{24}).$$

Then these matrices satisfy (cf. [26]):

Lemma 2.3.

$$\begin{aligned} SS^\dagger &= TT^\dagger = \mathrm{id}, \\ STS &= T^{-1} S T^{-1}, \\ S^2 &= \hat{C}, \\ T\hat{C} &= \hat{C} T = T, \end{aligned}$$

where $\hat{C}_{\lambda\mu} = \delta_{\lambda\bar{\mu}}$ is the conjugation matrix.

The above Lemma shows that S, T as defined there give rise to a representation of the modular group denoted by $\Gamma(1)$. This is the group generated by two matrices $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and the representation is given by $s \to S, t \to T$.

Let r be a rational number. Throughout this paper we will use T^r to denote a diagonal matrix whose (λ, λ) entry is given by $\exp(2\pi i (\Delta_\lambda - \frac{c_0}{24}) r)$.

Moreover

$$N^\nu_{\lambda\mu} = \sum_\delta \frac{S_{\lambda,\delta} S_{\mu,\delta} S^*_{\nu,\delta}}{S_{1,\delta}}. \tag{3}$$

is known as Verlinde formula.

Sometimes we will refer the S, T matrices as defined above as **genus 0 modular matrices of $\mathcal{A}$** since they are constructed from the fusion rules, monodromies and minimal indices which can be thought as genus 0 **chiral data** associated to a Conformal Field Theory.

Let c be the central charge associated with the projective representations of $\mathrm{Diff}(S^1)$ of the conformal net $\mathcal{A}$ (cf. [17]). Note that by [8] c is uniquely determined for a conformal net. We will see that c is always rational for a completely rational net (see (4) of Th. 4.5 for a more refined statement).

It is proved in Lemma 9.7 of [18] that $c_0 - c \in 4\mathbb{Z}$ for complete rational nets.

The commutative algebra generated by λ's with structure constants $N_{\lambda\mu}^{\nu}$ is called **fusion algebra** of $\mathcal{A}$. If Y is invertible, it follows from Lemma 2.3 and equation (3) that any nontrivial irreducible representation of the fusion algebra is of the form $\lambda \to \frac{S_{\lambda\mu}}{S_{1\mu}}$ for some μ.

2.4 The Galois action on Rehren's S, T matrices

The basic idea in the theory of the Galois action [6][7] is to look at the field F obtained by adjoining to the rationales $\mathbb{Q}$ the matrix elements of all modular transformations as defined after Lemma 2.3. One may show that, as a consequence of Verlinde's formula, F is a finite Abelian extension of $\mathbb{Q}$. By the theorem of Kronecker and Weber this means that F is a subfield of some cyclotomic field $\mathbb{Q}[\zeta_n]$ for some integer n, where $\zeta_n = \exp\left(\frac{2\pi i}{n}\right)$ is a primitive n-th root of unity. We'll call the conductor of $\mathcal{A}$ the smallest n for which $F \subseteq \mathbb{Q}[\zeta_n]$ and which is divisible by the order of the T matrix.

The above results imply that the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ is a homomorphic image of the Galois group $\mathcal{G}_n = \mathrm{Gal}(\mathbb{Q}[\zeta_n]/\mathbb{Q})$. But it is known that $\mathcal{G}_n$ is isomorphic to the group $(\mathbb{Z}/n\mathbb{Z})^*$ of prime residues modulo n, its elements being the Frobenius maps $\sigma_l : \mathbb{Q}[\zeta_n] \to [\zeta_n]$ that leave $\mathbb{Q}$ fixed, and send ζ_n to ζ_n^l for l coprime to n. Consequently, the maps σ_l are automorphisms of F over $\mathbb{Q}$.

According to [7], we have (for l coprime to the conductor)

$$\sigma_l\left(S_{\lambda,\mu}\right) = \varepsilon_l(\mu) S_{\lambda,\pi_l(\mu)} \tag{4}$$

for some permutation $\pi_l \in Sym(\mathcal{P})$ of the irreducible representations and some function $\varepsilon_l : \mathcal{P} \to \{-1,+1\}$. Upon introducing the orthogonal monomial matrices

$$\left(G_l\right)_{\lambda,\mu} = \varepsilon_l(\mu)\delta_{\lambda,\pi_l(\mu)} \tag{5}$$

and denoting by $\sigma_l(M)$ the matrix that one obtains by applying σ_l to M elementwise, we have

$$\sigma_l(S) = SG_l = G_l^{-1}S \tag{6}$$

Note that for l and m both coprime to the conductor

$$\begin{aligned} \pi_{lm} &= \pi_l \pi_m \\ G_{lm} &= G_l G_m \end{aligned}$$

The Galois action on T is given by

$$\sigma_l(T) = T^l \tag{7}$$

2.5 The orbifolds

Let $\mathcal{A}$ be an irreducible conformal net on a Hilbert space $\mathcal{H}$ and let Γ be a finite group. Let $V : \Gamma \to U(\mathcal{H})$ be a unitary representation of Γ on $\mathcal{H}$. If $V : \Gamma \to U(\mathcal{H})$ is not faithful, we set $\Gamma' := \Gamma/\mathrm{ker}V$.

Definition 2.4. *We say that Γ acts properly on $\mathcal{A}$ if the following conditions are satisfied:*

(1) For each fixed interval I and each $g \in \Gamma$, $\alpha_g(a) := V(g)aV(g^*) \in \mathcal{A}(I), \forall a \in \mathcal{A}(I)$;

(2) For each $g \in \Gamma$, $V(g)\Omega = \Omega, \forall g \in \Gamma$.

We note that if Γ acts properly, then $V(g)$, $g \in \Gamma$ commutes with the unitary representation U of $\mathsf{M\ddot{o}b}$.

Define $\mathcal{B}(I) := \{a \in \mathcal{A}(I) | \alpha_g(a) = a, \forall g \in \Gamma\}$ and $\mathcal{A}^\Gamma(I) := \mathcal{B}(I)P_0$ on $\mathcal{H}_0$ where $\mathcal{H}_0 := \{x \in \mathcal{H} | V(g)x = x, \forall g \in \Gamma\}$ and P_0 is the projection from $\mathcal{H}$ to $\mathcal{H}_0$. Then U restricts to an unitary representation (still denoted by U) of $\mathsf{M\ddot{o}b}$ on $\mathcal{H}_0$. Then:

Proposition 2.5. *The map $I \in \mathcal{I} \to \mathcal{A}^\Gamma(I)$ on $\mathcal{H}_0$ together with the unitary representation (still denoted by U) of $\mathsf{M\ddot{o}b}$ on $\mathcal{H}_0$ is an irreducible Möbius covariant net.*

The irreducible Möbius covariant net in Prop. 2.5 will be denoted by $\mathcal{A}^\Gamma$ and will be called the *orbifold of* $\mathcal{A}$ with respect to Γ. When Γ is generated by $h_1, ..., h_k$, we will write $\mathcal{A}^\Gamma$ as $\mathcal{A}^{\langle h_1, ..., h_k \rangle}$.

2.6 Complete rationality

We first recall some definitions from [19] . Recall that $\mathcal{I}$ denotes the set of intervals of S^1. Let $I_1, I_2 \in \mathcal{I}$. We say that I_1, I_2 are disjoint if $\bar{I}_1 \cap \bar{I}_2 = \varnothing$, where $\bar{I}$ is the closure of I in S^1. Denote by $\mathcal{I}_2$ the set of unions of disjoint 2 elements in $\mathcal{I}$. Let $\mathcal{A}$ be an irreducible Möbius covariant net as in §2.1. For $E = I_1 \cup I_2 \in \mathcal{I}_2$, let $I_3 \cup I_4$ be the interior of the complement of $I_1 \cup I_2$ in S^1 where I_3, I_4 are disjoint intervals. Let

$$\mathcal{A}(E) := A(I_1) \vee A(I_2), \quad \hat{A}(E) := (A(I_3) \vee A(I_4))'.$$

Note that $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. Recall that a net $\mathcal{A}$ is *split* if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is naturally isomorphic to the tensor product of von Neumann algebras $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ for any disjoint intervals $I_1, I_2 \in \mathcal{I}$. $\mathcal{A}$ is *strongly additive* if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ where $I_1 \cup I_2$ is obtained by removing an interior point from I.

Definition 2.6. *[19] $\mathcal{A}$ is said to be completely rational if $\mathcal{A}$ is split, strongly additive, and the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is finite for some $E \in \mathcal{I}_2$. The value of the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ (it is independent of E by Prop. 5 of [19]) is denoted by $\mu_{\mathcal{A}}$ and is called the μ-index of $\mathcal{A}$. If the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is infinity for some $E \in \mathcal{I}_2$, we define the μ-index of $\mathcal{A}$ to be infinity.*

A formula for the μ-index of a subnet is proved in [19]. With the result on strong additivity for $\mathcal{A}^\Gamma$ in [29], we have the complete rationality in following theorem.

Note that, by our recent results in [23], every irreducible, split, local conformal net with finite μ-index is automatically strongly additive.

Theorem 2.7. *Let $\mathcal{A}$ be an irreducible Möbius covariant net and let Γ be a finite group acting properly on $\mathcal{A}$. Suppose that $\mathcal{A}$ is completely rational. Then:*

(1): $\mathcal{A}^\Gamma$ is completely rational and $\mu_{\mathcal{A}^\Gamma} = |\Gamma'|^2 \mu_{\mathcal{A}}$;

(2): There are only a finite number of irreducible covariant representations of $\mathcal{A}^\Gamma$ (up to unitary equivalence), and they give rise to a unitary modular category as defined in II.5 of [27] by the construction as given in §1.7 of [30].

Suppose that $\mathcal{A}$ and Γ satisfy the assumptions of Th. 2.7. Then $\mathcal{A}^\Gamma$ has only finitely number of irreducible representations $\dot\lambda$ and

$$\sum_{\dot\lambda} d(\dot\lambda)^2 = \mu_{\mathcal{A}^\Gamma} = |\Gamma'|^2 \mu_{\mathcal{A}}.$$

The set of such $\dot\lambda$'s is closed under conjugation and compositions, and by Cor. 32 of [19], the Y-matrix in (1) for $\mathcal{A}^\Gamma$ is non-degenerate, and we will denote the corresponding genus 0 modular matrices by $\dot S, \dot T$. Denote by $\dot\lambda$ (resp. μ) the irreducible covariant representations of $\mathcal{A}^\Gamma$ (resp. $\mathcal{A}$) with finite index. Denote by $b_{\mu\dot\lambda} \in \mathbb{N} \cup \{0\}$ the multiplicity of representation $\dot\lambda$ which appears in the restriction of representation μ when restricting from $\mathcal{A}$ to $\mathcal{A}^\Gamma$. The $b_{\mu\dot\lambda}$ are also known as the *branching rules.* An irreducible covariant representation $\dot\lambda$ of $\mathcal{A}^\Gamma$ is called an *untwisted* representation if $b_{\mu\dot\lambda} \neq 0$ for some representation μ of $\mathcal{A}$. These are representations of $\mathcal{A}^\Gamma$ which appear as subrepresentations in the restriction of some representation of $\mathcal{A}$ to $\mathcal{A}^\Gamma$. A representation is called *twisted* if it is not untwisted.

2.7 Restriction to the real line: Solitons

Denote by $\mathcal{I}_0$ the set of open, connected, non-empty, proper subsets of $\mathbb{R}$, thus $I \in \mathcal{I}_0$ iff I is an open interval or half-line (by an interval of $\mathbb{R}$ we shall always mean a non-empty open bounded interval of $\mathbb{R}$).

Given a net $\mathcal{A}$ on S^1 we shall denote by $\mathcal{A}_0$ its restriction to $\mathbb{R} = S^1 \smallsetminus \{-1\}$. Thus $\mathcal{A}_0$ is an isotone map on $\mathcal{I}_0$, that we call a *net on* $\mathbb{R}$. In this paper we denote by $J_0 := (0, \infty) \subset \mathbb{R}$.

A representation π of $\mathcal{A}_0$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I}_0 \mapsto \pi_I$ that associates to each $I \in \mathcal{I}_0$ a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_{\tilde I} \restriction \mathcal{A}(I) = \pi_I, \quad I \subset \tilde I, \quad I, \tilde I \in \mathcal{I}_0 \ .$$

A representation π of $\mathcal{A}_0$ is also called a *soliton*. As $\mathcal{A}_0$ satisfies half-line duality, namely

$$\mathcal{A}_0(-\infty, a)' = \mathcal{A}_0(a, \infty), a \in \mathbb{R},$$

by the usual DHR argument [10] π is unitarily equivalent to a representation ρ which acts identically on $\mathcal{A}_0(-\infty,0)$, thus ρ restricts to an endomorphism of $\mathcal{A}(J_0) = \mathcal{A}_0(0,\infty)$. ρ is said to be localized on J_0 and we also refer to ρ as soliton endomorphism.

Clearly a representation π of $\mathcal{A}$ restricts to a soliton π_0 of $\mathcal{A}_0$. But a representation π_0 of $\mathcal{A}_0$ does not necessarily extend to a representation of $\mathcal{A}$.

If $\mathcal{A}$ is strongly additive, and a representation π_0 of $\mathcal{A}_0$ extends to a DHR representation of $\mathcal{A}$, then it is easy to see that such an extension is unique, and in this case we will use the same notation π_0 to denote the corresponding DHR representation of $\mathcal{A}$.

2.8 Induction and restriction for a net and its subnet

Let $\mathcal{A}$ be a Möbius covariant net. By a Möbius (resp. conformal) covariant subnet $\mathcal{B} \subset \mathcal{A}$ we mean a map

$$I \in \mathcal{I} \to \mathcal{B}(I) \subset \mathcal{A}(I)$$

that associates to each $I \in \mathcal{I}$ a von Neumann subalgebra $\mathcal{B}(I)$ so that isotony and covariance with respect to the Möbius (resp. conformal) group hold.

Given a bounded interval $I_0 \in \mathcal{I}_0$ we fix canonical endomorphism γ_{I_0} associated with $\mathcal{B}(I_0) \subset \mathcal{A}(I_0)$. Then we can choose for each $I \subset \mathcal{I}_0$ with $I \supset I_0$ a canonical endomorphism γ_I of $\mathcal{A}(I)$ into $\mathcal{B}(I)$ in such a way that $\gamma_I \restriction \mathcal{A}(I_0) = \gamma_{I_0}$ and γ_{I_1} is the identity on $\mathcal{B}(I_1)$ if $I_1 \in \mathcal{I}_0$ is disjoint from I_0, where $\gamma_I \equiv \gamma_I \restriction \mathcal{B}(I)$.

We then have an endomorphism γ of the C^*-algebra $\mathfrak{A} \equiv \overline{\cup_I \mathcal{A}(I)}$ (I bounded interval of $\mathbb{R}$).

Given a DHR endomorphism ρ of $\mathcal{B}$ localized in I_0, the induction α_ρ of ρ is the endomorphism of $\mathfrak{A}$ given by

$$\alpha_\rho \equiv \gamma^{-1} \cdot \mathrm{Ad}\varepsilon(\rho,\gamma) \cdot \rho \cdot \gamma \ ,$$

where ε denotes the right braiding unitary symmetry (there is another choice for α associated with the left braiding). α_ρ is localized in a right half-line containing I_0, namely α_ρ is the identity on $\mathcal{A}(I)$ if I is a bounded interval contained in the left complement of I_0 in $\mathbb{R}$. Up to unitarily equivalence, α_ρ is localizable in any right half-line thus α_ρ is normal on left half-lines, that is to say, for every $a \in \mathbb{R}$, α_ρ is normal on the C^*-algebra $\mathfrak{A}(-\infty,a) \equiv \overline{\cup_{I\subset(-\infty,a)}\mathcal{A}(I)}$ (I bounded interval of $\mathbb{R}$), namely $\alpha_\rho \restriction \mathfrak{A}(-\infty,a)$ extends to a normal morphism of $\mathcal{A}(-\infty,a)$. When there are several subnets involved, we will use notation $\alpha_\rho^{\mathcal{B}\to\mathcal{A}}$ introduced in §3 of [32] to indicate the net and subnet where we apply the induction.

2.9 Preliminaries on cyclic orbifolds

In the rest of this paper we assume that $\mathcal{A}$ is completely rational. $\mathcal{D} := \mathcal{A} \otimes \mathcal{A} ... \otimes \mathcal{A}$ (n-fold tensor product) and $\mathcal{B} := \mathcal{D}^{\mathbb{Z}_n}$ (resp. $\mathcal{D}^{\mathbb{P}_n}$ where $\mathbb{P}_n$ is the permutation group on n letters) is the fixed point subnet of $\mathcal{D}$ under the action of cyclic permutations

(resp. permutations). Recall that $J_0 = (0, \infty) \subset \mathbb{R}$. Note that the action of $\mathbb{Z}_n$ (resp. $\mathbb{P}_n$) on $\mathcal{D}$ is faithful and proper. Let $v \in \mathcal{D}(J_0)$ be a unitary such that $\beta_g(v) = e^{\frac{2\pi i}{n}} v$ (such v exists by P. 48 of [15]) where g is the generator of the cyclic group $\mathbb{Z}_n$ and β_g stands for the action of g on $\mathcal{D}$. Note that $\sigma := \mathrm{Ad}_v$ is a DHR representation of $\mathcal{B}$ localized on J_0. Let $\gamma : \mathcal{D}(J_0) \to \mathcal{B}(J_0)$ be the canonical endomorphism from $\mathcal{D}(J_0)$ to $\mathcal{B}(J_0)$ and let $\gamma_{\mathcal{B}} := \gamma \restriction \mathcal{B}(J_0)$. Note $[\gamma] = [1] + [g] + ... + [g^{n-1}]$ as sectors of $\mathcal{D}(J_0)$ and $[\gamma_{\mathcal{B}}] = [1] + [\sigma] + ... + [\sigma^{n-1}]$ as sectors of $\mathcal{B}(J_0)$. Here $[g^i]$ denotes the sector of $\mathcal{D}(J_0)$ which is the automorphism induced by g^i. All the sectors considered in the rest of this paper will be sectors of $\mathcal{D}(J_0)$ or $\mathcal{B}(J_0)$ as should be clear from their definitions. All DHR representations will be assumed to be localized on J_0 and have finite statistical dimensions unless noted otherwise. For simplicity of notations, for a DHR representation σ_0 of $\mathcal{D}$ or $\mathcal{B}$ localized on J_0, we will use the same notation σ_0 to denote its restriction to $\mathcal{D}(J_0)$ or $\mathcal{B}(J_0)$ and we will make no distinction between local and global intertwiners for DHR representations localized on J_0 since they are the same by the strong additivity of $\mathcal{D}$ and $\mathcal{B}$. The following is Lemma 8.3 of [23]:

Lemma 2.8. *Let μ be an irreducible DHR representation of $\mathcal{B}$. Let i be any integer. Then:*
(1) $G(\mu, \sigma^i) := \varepsilon(\mu, \sigma^i)\varepsilon(\sigma^i, \mu) \in \mathbb{C}$, $G(\mu, \sigma)^i = G(\mu, \sigma^i)$. Moreover $G(\mu, \sigma)^n = 1$;
(2) If $\mu_1 \prec \mu_2\mu_3$ with μ_1, μ_2, μ_3 irreducible, then $G(\mu_1, \sigma^i) = G(\mu_2, \sigma^i)G(\mu_3, \sigma^i)$;
(3) μ is untwisted if and only if $G(\mu, \sigma) = 1$;
(4) $G(\bar{\mu}, \sigma^i) = \bar{G}(\mu, \sigma^i)$.

2.10 One cycle case

First we recall the construction of solitons for permutation orbifolds in §6 of [23]. Let $h : S^1 \smallsetminus \{-1\} \simeq \mathbb{R} \to S^1$ be a smooth, orientation preserving, injective map which is smooth also at $\pm\infty$, namely the left and right limits $\lim_{z \to -1^{\pm}} \frac{\mathrm{d}^n h}{\mathrm{d}z^n}$ exist for all n.

The range $h(S^1 \smallsetminus \{-1\})$ is either S^1 minus a point or a (proper) interval of S^1.

With $I \in \mathcal{I}$, $-1 \notin I$, we set

$$\Phi_{h,I} \equiv \mathrm{Ad}U(k) \ ,$$

where $k \in \mathrm{Diff}(S^1)$ and $k(z) = h(z)$ for all $z \in I$ and U is the projective unitary representation of $\mathrm{Diff}(S^1)$ associated with $\mathcal{A}$. Then $\Phi_{h,I}$ does not depend on the choice of $k \in \mathrm{Diff}(S^1)$ and

$$\Phi_h : I \mapsto \Phi_{h,I}$$

is a well defined soliton of $\mathcal{A}_0 \equiv \mathcal{A} \restriction \mathbb{R}$.

Clearly $\Phi_h(\mathcal{A}_0(\mathbb{R}))'' = \mathcal{A}(h(S^1 \smallsetminus \{-1\}))''$, thus Φ_h is irreducible if the range of h is dense, otherwise it is a type III factor representation. It is easy to see that, in the last case, Φ_h does not depend on h up to unitary equivalence.

Let now $f : S^1 \to S^1$ be the degree n map $f(z) \equiv z^n$. There are n right inverses h_i, $i = 0, 1, \dots n-1$, for f (n-roots); namely there are n injective smooth maps

$h_i : S^1 \setminus \{-1\} \to S^1$ such that $f(h_i(z)) = z$, $z \in S^1 \setminus \{-1\}$. The h_i's are smooth also at $\pm\infty$.

Note that the ranges $h_i(S^1 \setminus \{-1\})$ are n pairwise disjoint intervals of S^1, thus we may fix the labels of the h_i's so that these intervals are counterclockwise ordered, namely we have $h_0(1) < h_1(1) < \cdots < h_{n-1}(1) < h_0(1)$, and we choose $h_j = e^{\frac{2\pi i j}{n}} h_0, 0 \leq j \leq n-1$. When no confusion arises, we will write h_0 simply as $z^{\frac{1}{n}}$ and $\Phi_{h_j,I}(x) = R_{\frac{2\pi j}{n}} R_{z^{\frac{1}{n}}}(x)$.

For any interval I of $\mathbb{R}$, we set

$$\pi_{1,\{0,1\ldots n-1\},I} \equiv \chi_I \cdot (\Phi_{h_0,I} \otimes \Phi_{h_1,I} \otimes \cdots \otimes \Phi_{h_{n-1},I}) \ , \tag{8}$$

where χ_I is the natural isomorphism from $\mathcal{A}(I_0)\otimes\cdots\otimes\mathcal{A}(I_{n-1})$ to $\mathcal{A}(I_0)\vee\cdots\vee\mathcal{A}(I_{n-1})$ given by the split property, with $I_k \equiv h_k(I)$. Clearly $\pi_{1,\{0,1\ldots n-1\}}$ is a soliton of $\mathcal{D}_0 \equiv \mathcal{A}_0 \otimes \mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_0$ (n-fold tensor product). Let $p \in \mathbb{P}_n$. We set

$$\pi_{1,\{p(0),p(1),\ldots,p(n-1)\}} = \pi_{1,\{0,1\ldots,n-1\}} \cdot \beta_{p^{-1}} \tag{9}$$

where β is the natural action of $\mathbb{P}_n$ on $\mathcal{D}$, and $\pi_{1,\{0,1\ldots,n-1\}}$ is as in (8). Let λ be a DHR representation of $\mathcal{A}$. Given an interval $I \subset S^1 \setminus \{-1\}$, we set

Definition 2.9.

$$\pi_{\lambda,\{p(0),p(1),\ldots,p(n-1)\},I}(x) = \pi_{\lambda,J}(\pi_{1,\{p(0),p(1),\ldots,p(n-1)\},I}(x)) \ , \quad x \in \mathcal{D}(I) \ ,$$

where $\pi_{1,\{p(0),p(1),\ldots,p(n-1)\},I}$ is defined as in (9), and J is any interval which contains $I_0 \cup I_1 \cup \ldots \cup I_{n-1}$. Denote the corresponding soliton by $\pi_{\lambda,\{p(0),p(1),\ldots,p(n-1)\}}$. When p is the identity element in $\mathbb{P}_n$, we will denote the corresponding soliton by $\pi_{\lambda,n}$.

2.11 Some properties of S matrix for general orbifolds

Let $\mathcal{A}$ be a completely rational conformal net and let Γ be a finite group acting properly on $\mathcal{A}$. By Th. 2.7 $\mathcal{A}^\Gamma$ has only finitely many irreducible representations. We use $\dot\lambda$ (resp. μ) to label representations of $\mathcal{A}^\Gamma$ (resp. $\mathcal{A}$). We will denote the corresponding genus 0 modular matrices by $\dot S, \dot T$. Denote by $\dot\lambda$ (resp. μ) the irreducible covariant representations of $\mathcal{A}^\Gamma$ (resp. $\mathcal{A}$) with finite index. Recall that $b_{\mu\dot\lambda} \in \mathbb{Z}$ denote the multiplicity of representation $\dot\lambda$ which appears in the restriction of representation μ when restricting from $\mathcal{A}$ to $\mathcal{A}^\Gamma$. $b_{\mu\dot\lambda}$ is also known as the branching rules. We have:

Lemma 2.10. *(1) If τ is an automorphism (i.e., $d(\tau) = 1$) then $S_{\tau(\lambda)\mu} = G_1(\tau,\mu)^* S_{\lambda\mu}$ where $\tau(\lambda) := \tau\lambda, G_1(\tau,\mu) = \epsilon(\tau,\mu)\epsilon(\mu,\tau)$;*

(2) For any $h \in \Gamma$, let $h(\lambda)$ be the DHR representation $\lambda \cdot \mathrm{Ad}_{h^{-1}}$. Then $S_{\lambda\mu} = S_{h(\lambda)h(\mu)}$;

(3) If $[\alpha_{\dot\lambda}] = [\mu\alpha_{\dot\delta}]$, then for any $\dot\lambda_1, \mu_1$ with $b_{\dot\lambda_1\mu_1} \neq 0$ we have $\frac{S_{\dot\lambda\dot\lambda_1}}{S_{\dot 1\dot\lambda_1}} = \frac{S_{\mu\mu_1}}{S_{1\mu_1}}\frac{S_{\dot\delta\dot\lambda_1}}{S_{\dot 1\dot\lambda_1}}$;

(4) We can choose $c_0(\mathcal{A}^\Gamma)$ so that $c_0(\mathcal{A}^\Gamma) = c_0(\mathcal{A})$ (cf. Definition 2.2).

Proof (1), (2) follows from Lemma 9.1 of [18]. (3) follows from the proof on Page 182 of [31] or Lemma 6.4 of [4]. ∎

2.12 Fusions of solitons in cyclic orbifolds

Let $\mathcal{B} \subset \mathcal{D}$ be as in section 2.9. We note that Th. 8.4 of [18] gives a list of all irreducible representations of $\mathcal{B}$.

Remark 2.11. *For simplicity we will label the representation $\pi_{\lambda,g^j,i}$ ($g = (01...n-1)$) by (λ, g^j, i), and when $i = 0$ (resp. $j = 0$) which stands for the trivial representation we will denote the corresponding representation simply as (λ, g^j) (resp. $(\lambda, 1)$). When 1 is used to denote the representation of a net, it will always be the vacuum representation.*

Lemma 2.12. *(1) $G(\sigma, (\mu, g)) = e^{\frac{2\pi i}{n}}$;*
(2) There exists an automorphism $\tau_{n,\mathcal{A}}, [\tau_{n,\mathcal{A}}^2] = [1]$ such that

$$S_{(\lambda,1),(\mu,g)} = \frac{1}{n} S_{\lambda,\tau_{n,\mathcal{A}}(\mu)}.$$

For simplicity we will denote $\tau_{n,\mathcal{A}}$ simply as τ_n when the underlying net $\mathcal{A}$ is clear.

Proof (1) follows from remark 4.18 in [9], and (2) follows from Lemma 9.3 of [18]. ■

Remark 2.13. *By (4) of Lemma 2.10, we can choose $c_0(\mathcal{B})$ so that $c_0(\mathcal{B}) = nc_0(\mathcal{A})$. We will make such choice in the rest of this paper.*

3 Squares of conformal nets

Definition 3.1. *Let $\mathcal{A}_i, 1 \leq i \leq 4$ be four Möbius covariant nets such that $A_3 \subset A_2, A_2 \subset A_1, A_3 \subset A_4$ and $A_4 \subset A_1$ are subnets. Then the square*
$$\begin{array}{ccc} A_2 & \subset & A_1 \\ \bigcup & & \bigcup \\ A_3 & \subset & A_4 \end{array}$$
is called a square of Möbius covariant nets.

Let $N = nk, g = (123...N), \mathcal{D} := \mathcal{A} \otimes \mathcal{A} \otimes ...\mathcal{A}$. Then g^n is n product of k cycles $g_1, ..., g_n$, with $g_{i+1} = (i(i+n)(i+2n)...(i+(k-1)n)), 0 \leq i \leq n-1$. The following square of conformal nets play an important role in this paper:

$$\begin{array}{ccc} \mathcal{D}^{\langle g \rangle} & \subset & \mathcal{D}^{\langle g^n \rangle} \\ \bigcup & & \bigcup \\ \mathcal{B}_1 := \mathcal{D}^{\langle g, g_1, ..., g_n \rangle} & \subset & \mathcal{B}_2 := \mathcal{D}^{\langle g_1, ..., g_n \rangle} \end{array}$$

Proposition 3.2. *(1) We identify $\mathcal{D}^{\langle g_1,...,g_n \rangle}$ with n tensor products of a k-th order cyclic permutation orbifold $(\mathcal{A} \otimes ... \otimes \mathcal{A})^{\langle h_1 \rangle}$ in a natural way. Then $\mathcal{D}^{\langle g, g_1,...,g_n \rangle} \subset \mathcal{D}^{\langle g_1,...,g_n \rangle}$ is a cyclic permutation of order n; Denote by h_2 the cyclic permutation on $\mathcal{D}^{\langle g_1,...,g_n \rangle}$ which comes from permutation $(01...(n-1))(n(n+1)...(n+n-1))...((k-1)n...((k-1)n+n-1))$ of $\mathcal{D}$;*
(2)$\alpha^{\mathcal{B}_1 \to \mathcal{D}^{\langle g \rangle}}_{((\lambda,h_1),h_2)} \succ (\lambda, g)$;
(3)$\alpha^{\mathcal{B}_1 \to \mathcal{D}^{\langle g \rangle}}_{((\lambda,h_1^i),1)} = (\lambda, g^{ni})$;
(4) When $(k, n) = 1$, $\alpha^{\mathcal{B}_1 \to \mathcal{D}^{\langle g \rangle}}_{((\lambda,1),h_2^k)} \succ (\lambda, g^k)$.

Proof (1) follows from definition. As for (2), we first show that (λ, g) and $((\lambda, h_1), h_2)$ come from the restriction of the same soliton of $\mathcal{D}$. This can be seen from definition 2.9 as follows: (λ, g) comes from a soliton of $\mathcal{D}$ defined by:

$$x_0 \otimes x_1 \otimes ... \otimes x_N \in \mathcal{A}(I) \otimes ... \mathcal{A}(I) \to \pi_\lambda(R_{z^{\frac{1}{N}}}(x_0) \vee R_{\frac{2\pi}{N}} R_{z^{\frac{1}{N}}}(x_1) \vee ... R_{\frac{2\pi(N-1)}{N}} R_{z^{\frac{1}{N}}}(x_{N-1}))$$

Let $y_i = x_i \otimes x_{n+i} \otimes ... \otimes x_{n(k-1)+i}, 0 \leq i \leq n-1$. Then $((\lambda, h_1), h_2)$ comes from a soliton of $\mathcal{D}$ defined by

$$\begin{aligned} y_0 \otimes y_1 \otimes ... \otimes y_{n-1} \to & \pi_{\lambda, h_1}(R_{z^{\frac{1}{n}}}(y_0) \vee R_{\frac{2\pi}{n}} R_{z^{\frac{1}{n}}}(y_1) \vee ... R_{\frac{2\pi(n-1)}{n}} R_{z^{\frac{1}{n}}}(y_{n-1}) \\ & = \pi_\lambda(R_{z^{\frac{1}{N}}}(x_0) \vee R_{\frac{2\pi}{N}} R_{z^{\frac{1}{N}}}(x_1) \vee ... R_{\frac{2\pi(N-1)}{N}} R_{z^{\frac{1}{N}}}(x_{N-1})) \end{aligned}$$

where we have used

$$R_{z^{\frac{1}{k}}} R_{\frac{2\pi i}{n}}(x) = R_{\frac{2\pi i}{kn}} R_{z^{\frac{1}{k}}}(x), R_{z^{\frac{1}{k}}} R_{z^{\frac{1}{n}}}(x) = R_{z^{\frac{1}{nk}}}(x), \forall x \in \mathcal{A}(I)$$

Now by Th. 4.8 of [18], (λ, g) is the component of the above soliton where g acts trivially, and $((\lambda, h_1), h_2)$ is the component of the same soliton where $\langle g, g_1, ..., g_n \rangle$ acts trivially. It follow that the restriction of (λ, g) to $\mathcal{B}_1$ contains $((\lambda, h_1), h_2)$, and (2) is proved.

To prove (3), we first show that $\alpha^{\mathcal{B}_1 \to \mathcal{D}^{\langle g \rangle}}_{((\lambda, h_1^i), 1)} \succ (\lambda, g^{ni})$. As in (2) it is sufficient to show that $((\lambda, h_1^i), 1), (\lambda, g^{ni})$ come from restrictions of the same soliton of $\mathcal{D}$, and as in (2) this follows by definition. By using the index formula in Th. 4.5 and (3) of Prop. 7.4 of [18], we have $d(((\lambda, h_1^i), 1)) = d((\lambda, g^{ni}))$, and (3) is proved. (4) is proved in a similar way as in (2): we check that $\alpha^{\mathcal{B}_1 \to \mathcal{D}^{\langle g \rangle}}_{((\lambda, 1), h_2^k)} \succ (\lambda, g^k)$ by showing that $((\lambda, 1), h_2^k), (\lambda, g^k)$ come from the same soliton of $\mathcal{D}$. By definition 2.9 $((\lambda, 1), h_2^k)$ comes from a soliton of $\mathcal{D}$ defined by

$$\begin{aligned} x_0 \otimes x_1 \otimes ... \otimes x_N \in \mathcal{A}(I) \otimes ... \mathcal{A}(I) \to & \pi_\lambda(R_{z^{\frac{1}{n}}}(x_0) \vee R_{\frac{2\pi}{n}} R_{z^{\frac{1}{n}}}(x_{-k}) \vee ... \\ & R_{\frac{2\pi(n-1)}{n}} R_{z^{\frac{1}{n}}}(x_{-k(n-1)})) \otimes \\ & \pi_\lambda(R_{z^{\frac{1}{n}}}(x_{-1}) \vee R_{\frac{2\pi}{n}} R_{z^{\frac{1}{n}}}(x_{-1-k}) \vee ... \\ & R_{\frac{2\pi(n-1)}{n}} R_{z^{\frac{1}{n}}}(x_{-1-k(n-1)})) \otimes \\ & ... \otimes \pi_\lambda(R_{z^{\frac{1}{n}}}(x_{-k+1}) \vee \\ & R_{\frac{2\pi}{n}} R_{z^{\frac{1}{n}}}(x_{-k+1-k}) \vee ... \\ & R_{\frac{2\pi(n-1)}{n}} R_{z^{\frac{1}{n}}}(x_{-k+1-k(n-1)})) \end{aligned}$$

where indices are defined modulo N. Let $y_{ki} = x_{ki} \otimes x_{n+ki} \otimes ... \otimes x_{n(k-1)+ki}, 0 \leq i \leq n-1$. Since $(n, k) = 1$, (λ, g^k) comes from a soliton of $\mathcal{D}^{\langle g_1, \cdots g_n \rangle}$ defined by

$$\begin{aligned} \pi_{\lambda,(0,k,2k,...,k(n-1))}(y_0 \otimes y_1 \otimes ... \otimes y_{n-1}) &= \pi_{\lambda,(0,1,2,...,(n-1))}(y_0 \otimes y_{-k} \otimes ... \otimes y_{-k(n-1)}) \\ &= \pi_\lambda(R_{z^{\frac{1}{n}}}(y_0) R_{\frac{2\pi}{n}} R_{z^{\frac{1}{n}}}(y_{-k}) \vee ... R_{\frac{2\pi(n-1)}{n}} R_{z^{\frac{1}{n}}}(y_{-(n-1)k})) \end{aligned}$$

where the indexes are defined modulo n. Then the soliton of $\mathcal{D}^{\langle g_1,\cdots g_n\rangle}$ above comes from restriction of soliton of $\mathcal{D}$ defined by

$$\begin{aligned} x_0 \otimes x_1 \otimes ... \otimes x_{N-1} \to & \pi_\lambda(R_{z^{\frac{1}{n}}}(x_0) \vee R_{\frac{2\pi}{n}} R_{z^{\frac{1}{n}}}(x_{-k}) \vee ...R_{\frac{2\pi(n-1)}{n}} R_{z^{\frac{1}{n}}}(x_{-k(n-1)}))\otimes \\ & \pi_\lambda(R_{z^{\frac{1}{n}}}(x_{-n}) \vee R_{\frac{2\pi}{n}} R_{z^{\frac{1}{n}}}(x_{-n-k}) \vee ...R_{\frac{2\pi(n-1)}{n}} R_{z^{\frac{1}{n}}}(x_{-n-k(n-1)}))\otimes \\ & ... \otimes \pi_\lambda(R_{z^{\frac{1}{n}}}(x_{n(-k+1)})\vee \\ & R_{\frac{2\pi}{n}} R_{z^{\frac{1}{n}}}(x_{n(-k+1)-k}) \vee ...R_{\frac{2\pi(n-1)}{n}} R_{z^{\frac{1}{n}}}(x_{n(-k+1)-k(n-1)})) \end{aligned}$$

which up to unitary equivalence (the unitary is a permutation of the tensor factors in the Hilbert space) is the same as the soliton defined at the beginning of the proof of (4). Thus we have shown that

$$\alpha^{\mathcal{B}_1 \to \mathcal{D}^{\langle g \rangle}}_{((\lambda,1),h_2^k)} \succ (\lambda, g^k).$$

3.1 Constraints on certain automorphisms

For simplicity of notations we define $\tau_{k,n} := \tau_{n,(\mathcal{A}\otimes\mathcal{A}\otimes...\otimes\mathcal{A})^{\mathbb{Z}_k}}$.

Proposition 3.3. *(1)$\tau_{n,\mathcal{A}\otimes\mathcal{A}...\otimes\mathcal{A}} = \tau_{n,\mathcal{A}} \otimes \tau_{n,\mathcal{A}} ... \otimes \tau_{n,\mathcal{A}}$ (k tensors);*
(2)$\tau_{k,n} = (\tau_n, 1, j_{k,n})$ with $k|2j_{k,n}$.

Proof Ad (1): Consider inclusions of sunbets $\mathcal{B}_2 \subset \mathcal{D}^{\langle g^n \rangle} \subset \mathcal{D}$. Note that by definition

$$\alpha^{\mathcal{B}_2 \to \mathcal{D}^{\langle g^n \rangle}}_{(\lambda_1,g_1)\otimes(\lambda_2,g_2)\otimes...\otimes(\lambda_n,g_n)} = (\lambda_1 \otimes \lambda_2 ... \otimes \lambda_n, g^n)$$

By Lemma 2.10 we have

$$\frac{S_{(\lambda_1,g_1)\otimes(\lambda_2,g_2)\otimes...\otimes(\lambda_n,g_n),(\mu_1,1)\otimes(\mu_2,1)\otimes...\otimes(\mu_n,1)}}{S_{1\otimes...\otimes 1,(\mu_1,1)\otimes...\otimes(\mu_n,1)}} = \frac{S_{(\lambda_1\otimes...\otimes\lambda_n,g^n),(\mu_1\otimes...\mu_n,1)}}{S_{1\otimes...\otimes 1,\mu_1\otimes...\otimes\mu_n}}$$

By Lemma 2.12 it follows that

$$S_{(\tau_{k,\mathcal{A}\otimes...\otimes\mathcal{A}}(\lambda_1\otimes...\otimes\lambda_n)),\mu_1\otimes...\otimes\mu_n} = S_{(\tau_{k,\mathcal{A}}\lambda_1\otimes...\otimes\tau_{k,\mathcal{A}}\lambda_n),\mu_1\otimes...\otimes\mu_n}$$

By unitarity of S matrix, and by replacing k with n, (1) is proved.

Ad (2): it is sufficient to show that $\alpha_{\tau_{k,n}} = \tau_n$, where the induction is with respect to the k-th cyclic permutation orbifold $(\mathcal{A}\otimes\mathcal{A}...\otimes\mathcal{A})^{\mathbb{Z}_k}$ and $\mathcal{A}\otimes\mathcal{A}...\otimes\mathcal{A}$ (k tensors). First we note that since $d(\tau_{k,n}) = 1$, and any twisted representation of $(\mathcal{A}\otimes\mathcal{A}...\otimes\mathcal{A})^{\mathbb{Z}_k}$ has index greater or equal to k^2 by Th. 4.5 and Prop. 7.4 of [18], it follows that $\alpha_{\tau_{k,n}} = \beta$ is a DHR representation of $\mathcal{A} \otimes \mathcal{A} ... \otimes \mathcal{A}$ (k tensors). Consider the square of nets:

$$\begin{array}{ccc} \mathcal{D}^{\langle h \rangle} & \subset & \mathcal{D} \\ \bigcup & & \bigcup \\ \mathcal{B}_1 = \mathcal{D}^{\langle g,g_1,\dots,g_n\rangle} & \subset & \mathcal{B}_2 := \mathcal{D}^{\langle g^n \rangle} \end{array}$$

where

$$h = (012...n-1)(n(n+1)...(n+n-1))...(((k-1)n)((k-1)n+1)...((k-1)n+n-1))$$

By definition $\alpha^{\mathcal{B}_1\to\mathcal{D}^{\langle h\rangle}}_{((\lambda,1),h_2)}=(\lambda,h)$ where by a slight abuse of notations we use λ to denote an irreducible DHR representation of $\mathcal{A}\otimes...\otimes\mathcal{A}$ (k tensors). By using (1), Lemma 2.12 and Lemma 2.10 we have $S_{\beta\lambda,\mu}=S_{\tau_n\lambda,\mu}$ for all λ,μ. By unitarity of S matrix and the fact that $[\tau^2_{k,n}]=[1]$ (2) is proved. ■

Proposition 3.4. *(1)$\tau_{k,n}(\lambda,h_1)=(\tau_N\tau_k\lambda,h_1,j)$ for some $0\leq j\leq k-1$;*

(2)τ_N is the vacuum if N is odd, and $\tau_N=\tau_2$ is N is even;

(3) $\tau_{k,n}$ is the vacuum if n is odd, and $j_{k,n}$ as in Prop. 3.3 is 0 modulo k if k is odd.

Proof Ad (1): By Prop. 3.3 we can assume that $\tau_{k,n}(\lambda,h_1)=(\mu,h_1,j)$. By Prop. 3.2 and Lemma 2.10 we have $S_{(\lambda,g)(\lambda_1,1)}=S_{\tau_{k,n}(\lambda,h_1),(\lambda_1,1)}$, hence $S_{\tau_k\mu,\lambda_1}=S_{\tau_N\lambda,\lambda_1}$ by (2) of Lemma 2.12. By unitarity of S matrix, (1) is proved.

Ad (2): By (1) we have $\alpha_{\tau_{k,n}(\lambda,h_1)}=\alpha_{(\tau_N\tau_k\lambda,h_1,j)}$, where the induction is with respect to the k-th order cyclic permutation orbifold of $\mathcal{A}\otimes\mathcal{A}\otimes...\otimes\mathcal{A}$ (k tensors). Note that $\alpha_{\tau_{k,n}}=(\tau_n,...,\tau_n)$ by Prop. 3.3. It follows by Th. 8.6 of [23] that $\tau_n^k=\tau_N\tau_k$. Hence if k is even, $\tau_N=\tau_k=\tau_2$, and if k is odd, $\tau_{nk}=\tau_n\tau_k$. Choose n even we have τ_k is the vacuum when k is odd.

Ad (3): (3) follows from (2) and (2) of Prop. 3.3 ■

Remark 3.5. *We can actually show that $\zeta_k^{j_{k,n}}=\zeta_2^{j_{2,2}}$ when k,n are even integers, but this fact will not be used in the paper.*

Theorem 3.6.

$$[\pi^n_{1,\{0,1,...,n-1\}}]=\bigoplus_{\lambda_1,...,\lambda_n}M_{\lambda_1,...,\lambda_n}[(\lambda_1,...,\lambda_n)]$$

where $M_{\lambda_1,...,\lambda_n}:=\sum_\lambda S^{2-2g}_{1,\lambda}\prod_{1\leq i\leq n}\frac{S_{\lambda_i,\lambda}}{S_{1,\lambda}}$ with $g=\frac{(n-1)(n-2)}{2}$, and $\pi_{1,\{0,1,...,n-1\}}$ is the soliton defined in equation (9).

Proof This is proved in Prop. 9.4 of [18] under the assumption $\tau_n^n=1$. The assumption follows by Prop. 3.4. ■

We note that the above Theorem was conjectured on Page 759 of [18] as a consequence of another conjecture on Page 758 of [18] which states that τ_N is the vacuum for all N. By Prop. 3.4 it is now enough to prove that τ_2 is the vacuum.

3.2 Properties of certain matrices

In this section we define and examine properties of certain matrices motivated by P.Bantay's Λ matrices in [1] and [2] which we recalled here for comparison. See [1] and [2] for more details. Suppose that a representation of the modular group $\Gamma(1)$ has been given. Let $r=\frac{k}{n}$ be a rational number in reduced form, i.e. with $n>0$ and

k and n coprime. Choose integers x and y such that $kx - ny = 1$, and define $r^* = \frac{x}{n}$. Then $m = \begin{pmatrix} k & y \\ n & x \end{pmatrix}$ belongs to $\Gamma(1)$, and we define the matrix $\Lambda\,(r)$ via

$$\Lambda\,(r)_{p,q} = \omega_p^{-r} M_{p,q} \omega_q^{-r^*}$$

One should fix some definite branch of the logarithm to make the above definition meaningful, but different choices lead to equivalent results. See the remark after Lemma 2.3 for our choice for genus 0 modular matrices.

It is a simple matter to show that $\Lambda(r)$ is well defined, i.e. does not depend on the actual choice of x and y, and $\Lambda(r)$ is periodic in r with period 1, i.e.

$$\Lambda(r+1) = \Lambda(r)$$

For $r = 0$ we just get back the S matrix

$$\Lambda(0) = S$$

and for a positive integer n we have

$$\Lambda\left(\frac{1}{n}\right) = T^{-\frac{1}{n}} S^{-1} T^{-n} S T^{-\frac{1}{n}} \tag{10}$$

Finally, we have

$$\Lambda\,(r^*)_p^q = \Lambda(r)_q^p$$
$$\Lambda\,(-r)_p^q = \overline{\Lambda\,(r)_p^q}$$

and the functional equation

$$\Lambda(\frac{-1}{r}) = T^{\frac{1}{r}} S T^r \Lambda(r) T^{\hat{r}} \tag{11}$$

where $\hat{r} = \frac{1}{kn}$.

Definition 3.7. *When $(i, N) = 1$, we define*

$$\widehat{\Lambda}_{\lambda_1,\lambda_2}(\frac{i}{N}) = N S_{(\lambda_1,g),(\lambda_2,g^i)}, \widehat{\Lambda}_{\lambda_1,\lambda_2}(r) = N S_{\lambda_1,\lambda_2}$$

where r is any integer.

We note that it follows from the definition that

$$\widehat{\Lambda}_{\lambda_1,\lambda_2}(\frac{i}{N}) = \widehat{\Lambda}_{\lambda_1,\lambda_2}(\frac{i}{N} + 1).$$

Proposition 3.8. *(1)$S_{(\lambda_1,g^{i_1},j_1),(\lambda_2,g^{i_2},j_2)} = \zeta_N^{-i_1 j_2 - i_2 j_1} S_{(\lambda_1,g^{i_1}),(\lambda_2,g^{i_2})}$;*

(2) If $(i_1, N) = 1$ and $\hat{i_1} i_1 \equiv 1 mod N$, then $S_{(\lambda_1,g^{i_1}),(\lambda_2,g^{i_2})} = \frac{1}{N}\widehat{\Lambda}_{\lambda_1,\lambda_2}(\frac{i_2 \hat{i_1}}{N})$;

(3) $\widehat{\Lambda}_{\lambda_1,\lambda_2}(r) = \widehat{\Lambda}_{\lambda_2,\lambda_1}(r^*)$;

(4) $\widehat{\Lambda}_{\lambda_1,\lambda_2}(1-r) = \overline{\widehat{\Lambda}_{\bar{\lambda}_1,\lambda_2}(r)}$;

Proof (1) follows from Lemma 2.10 and 2.8. For (2), let $h \in S_N$ so that $hg^{i_1}h^{-1} = g$. Then $hgh^{-1} = g^{\hat{i}_1}$. By Lemma 2.10 and definition 3.7, we have

$$S_{(\lambda_1, g^{i_1}),(\lambda_2, g^{i_2})} = S_{(\lambda_1, g),(\lambda_1, g^{i_1 \hat{i}_2})} = \frac{1}{N}\widehat{\Lambda}_{\lambda_1,\lambda_2}(\frac{i_2\widehat{i_1}}{N})$$

For (3), let $h' \in S_N$ be such that $h'g^ih'^{-1} = g$. Then $h'gh'^{-1} = g^{\hat{i}}$, and by Lemma 2.10 and the fact that S is symmetric we have

$$S_{(\lambda_1, g^i),(\lambda_2, g)} = S_{(\lambda_1, g),(\lambda_2, g^{\hat{i}})} = S_{(\lambda_2, g^{\hat{i}}),(\lambda_1, g)}$$

This proves (3) by definition.

For (4), by Prop. 6.1 of [23] the conjugate of (λ_1, g^i) is $(\bar{\lambda}_1, g^{-i})$. (4) now follows from definition and the property of S matrix under conjugation. ■

Proposition 3.9. $S_{(\lambda_1,g),(\lambda_2,g^{ni})} = S_{((\lambda_1,h_1),h_2),((\lambda_2,h_1^i),1)}$

Proof Consider the square of nets $\begin{array}{ccc} \mathcal{D}^{\langle g \rangle} & \subset & \mathcal{D}^{\langle g^n \rangle} \\ \bigcup & & \bigcup \\ \mathcal{B}_1 := \mathcal{D}^{\langle g, g_1, \ldots, g_n \rangle} & \subset & \mathcal{B}_2 := \mathcal{D}^{\langle g_1, \ldots, g_n \rangle} \end{array}$ By Prop. 3.2 and Lemma 2.10 we have

$$\frac{S_{((\lambda_2,h_1^i),1),((\lambda_1,h_1),h_2)}}{S_{1,((\lambda_1,h_1),h_2)}} = \frac{S_{(\lambda_2,g^{ni}),(\lambda_1,g)}}{S_{1,(\lambda_1,g)}}$$

But

$$\frac{S_{1,((\lambda_1,h_1),h_2)}}{S_{11}^{\mathcal{B}_1}} = d(((\lambda_1, h_1), h_2)) = d(\lambda_1)k^{n-1}\mu_{\mathcal{A}}^{\frac{(kn-1)}{2}}$$

and

$$\frac{S_{1,(\lambda_1,g)}}{S_{11}^{\mathcal{D}^{\langle g \rangle}}} = d(\lambda_1)\mu_{\mathcal{A}}^{\frac{(kn-1)}{2}}$$

where we have used Th. 4.5 and (3) of Prop. 7.4 in [18] in the calculation above. On the other hand

$$\frac{1}{S_{11}^{\mathcal{B}_1}} = \sqrt{\mu_{\mathcal{B}_1}} = k^n n\sqrt{\mu_{\mathcal{A}}}, \frac{1}{S_{11}^{\mathcal{D}^{\langle g \rangle}}} = \sqrt{\mu_{\mathcal{D}}^{\langle g \rangle}} = N\sqrt{\mu_{\mathcal{A}}},$$

and using these equations we obtain

$$S_{(\lambda_1,g),(\lambda_2,g^{ni})} = S_{((\lambda_1,h_1),1),((\lambda_2,1),h_2)}$$

■

Proposition 3.10. *Assume that* $(k, n) = 1$ *and* $N = kn$. *Then* $S_{(\lambda_1,g^n),(\lambda_2,g^k)} = \frac{1}{N}S_{\tau_k\lambda_1,\tau_n\lambda_2}$.

Proof Consider the square of nets $\begin{array}{ccc} \mathcal{D}^{\langle g\rangle} & \subset & \mathcal{D}^{\langle g^n\rangle} \\ \bigcup & & \bigcup \\ \mathcal{B}_1 := \mathcal{D}^{\langle g,g_1,\ldots,g_k\rangle} & \subset & \mathcal{B}_2 := \mathcal{D}^{\langle g_1,\ldots,g_k\rangle} \end{array}$ By Prop. 3.2 and Lemma 2.10 we have

$$\frac{S_{((\lambda_1,h_1),1),((\lambda_2,1),h_2)}}{S_{1,((\lambda_2,1),h_2)}} = \frac{S_{(\lambda_1,g^n),(\lambda_2,g^k)}}{S_{1,(\lambda_2,g^k)}}$$

But
$$\frac{S_{1,((\lambda_2,1),h_2)}}{S_{11}^{\mathcal{B}_1}} = d(((\lambda_2,1),h_2)) = d(\lambda_2)^k k^{n-1}\mu_{\mathcal{A}}^{\frac{k(n-1)}{2}}$$

and
$$\frac{S_{1,(\lambda_2,g^k)}}{S_{11}^{\mathcal{D}^{\langle g\rangle}}} = d(\lambda_2)^k \mu_{\mathcal{A}}^{\frac{k(n-1)}{2}}$$

where we have used Th. 4.5 and (3) of Prop. 7.4 in [18] in the calculation above. On the other hand

$$\frac{1}{S_{11}^{\mathcal{B}_1}} = \sqrt{\mu_{\mathcal{B}_1}} = k^n n\sqrt{\mu_{\mathcal{A}}}, \frac{1}{S_{11}^{\mathcal{D}^{\langle g\rangle}}} = \sqrt{\mu_{\mathcal{D}}^{\langle g\rangle}} = N\sqrt{\mu_{\mathcal{A}}},$$

and using these equations we obtain

$$S_{(\lambda_1,g^n),(\lambda_2,g^k)} = S_{((\lambda_1,h_1),1),((\lambda_2,1),h_2)} = \frac{1}{n}S_{(\lambda_1,h_1),(\tau_{k,n}(\lambda_2,1))}$$

Since $(k,n)=1$, $\zeta_k^{j_{k,n}} = 1$ by Prop. 3.4, and we have

$$\frac{1}{n}S_{(\lambda_1,h_1),(\tau_{k,n}(\lambda_2,1))} = \frac{1}{n}\frac{S_{(\lambda_1,h_1),\tau_{k,n}}}{S_{(\lambda_1,h_1),1}}\frac{1}{k}S_{\tau_k\lambda_1,\lambda_2} = \frac{1}{N}S_{\tau_k\lambda_1,\tau_n\lambda_2}$$

■

To prepare the statement of the main theorem in this section, we define

Definition 3.11. *Let $(k,n)=1$. Define a function g with value in Q mod $\mathbb{Z}$ by be the following equations:*

$$g(0)=1, g(\frac{k}{n}) = g(\frac{k}{n}\pm 1), g(\frac{k}{n}) + g(\frac{n}{k}) = \frac{-2\pi i}{24}(c-c_0)(3nk - \frac{n^2+k^2+1}{nk})$$

where c is the central charge of $\mathcal{A}$ and c_0 is as in definition 2.2.

Such a function is clearly uniquely determined by the defining equations. We will give further properties of g in Prop. 4.7. Let Λ be Bantay's Λ matrices as reviewed at the beginning of this section associated with genus 0 S,T matrices as defined after definition 2.2. Then we have:

Theorem 3.12.

$$\widehat{\Lambda}_{\lambda_1,\lambda_2}(r) = \exp(2\pi i g(r))\Lambda_{\lambda_1,\lambda_2}(r)$$

where $r \in \mathbb{Q}$ and $g(r)$ is as in definition 3.11.

Proof The idea is to consider equation $S = TSTST$ for the net $\mathcal{D}^{\langle g\rangle}$ with the order of g equal to nk and $(n,k) = 1$.

Let us compute the $(\lambda_1, g^n), (\lambda_2, g^{k-n})$ entry on both sides of the equation. Since $(k,n) = 1$, $(k-n, kn) = 1$. Let x_1, x_2 be integers such that $x_1(k-n) + knx_2 = 1$.

By Prop. 3.9 the left hand side is

$$S_{(\lambda_1,g^n),(\lambda_2,g^{k-n})} = S_{(\lambda_1,g^{nx_1}),(\lambda_2,g)} = S_{((\lambda_1,h_1^{x_1}),1),((\lambda_2,h_1),h_2)}$$

By Lemma 2.12 and Prop. 3.4 we have

$$S_{((\lambda_1,h_1^{x_1}),1),((\lambda_2,h_1),h_2)} = \frac{1}{n} S_{(\lambda_1,h_1^{x_1}),\tau_{k,n}(\lambda_2,h_1)} = \frac{1}{N}\frac{S_{\tau_k(\lambda_1),\tau_n}}{S_{\tau_k(\lambda_1),1}}\widehat{\Lambda}_{\lambda_1\lambda_2}(\frac{k-n}{k})$$

By (14) of [23] and remark 2.13 we have

$$T_{\lambda_1,g^n} = T_{\lambda_1}^{\frac{n}{k}}\exp(2\pi i(\frac{(k^2-1)(c-c_0)}{24k}), T_{\lambda_3,g^k} = T_{\lambda_3}^{\frac{k}{n}}\exp(2\pi i(\frac{(n^2-1)(c-c_0)}{24n}))$$

and

$$T_{\lambda_2,g^{k-n}} = T_{\lambda_2}^{\frac{1}{nk}}\exp(2\pi i(\frac{(n^2k^2-1)(c-c_0)}{24nk}))$$

By using the above equations and Prop. 3.10 we obtain the $(\lambda_1, g^n), (\lambda_2, g^{k-n})$ entry on the RHS is given by

$$\frac{1}{N}\sum_{\lambda_3}\exp(\frac{2\pi i(c-c_0)(3nk - \frac{n^2+k^2+1}{nk})}{24})T_{\lambda_1}S_{\tau_k(\lambda_1),\tau_n(\lambda_3)}T_{\lambda_3}^{\frac{k}{n}}\widehat{\Lambda}_{\lambda_3,\lambda_2}(\frac{k-n}{n})\frac{S_{\tau_k,\tau_n(\lambda_3)}}{S_{1\lambda_3}}T_{\lambda_2}^{\frac{1}{kn}}$$

Since $(k,n) = 1$, by Prop. 3.4 we have

$$\frac{S_{\tau_k,\lambda_3}}{S_{1,\lambda_3}} = \frac{S_{\tau_k,\tau_n\lambda_3}}{S_{1,\lambda_3}}$$

when k is odd, and when k is even, n must be odd and the above equation also holds. When comparing both the LHS and RHS, we see that the τ dependence canceled out from both sides and we are left with the following equation for $\widehat{\Lambda}$:

$$\widehat{\Lambda}_{\lambda_1,\lambda_2}(\frac{k-n}{k}) = \exp(\frac{2\pi i(c-c_0)(3nk - \frac{n^2+k^2+1}{nk})}{24})T_{\lambda_1}S_{\lambda_1,\lambda_3}T_{\lambda_3}^{\frac{k}{n}}\widehat{\Lambda}_{\lambda_3,\lambda_2}(\frac{k-n}{n})T_{\lambda_2}^{\frac{1}{kn}}$$

Comparing with the equation (11) of Bantay's Λ matrices and using Prop. 3.8 we conclude that there is a mod$\mathbb{Z}$ valued function as defined in definition 3.11 such that

$$\widehat{\Lambda}_{\lambda_1,\lambda_2}(r) = \exp(2\pi i g(r))\Lambda_{\lambda_1,\lambda_2}(r).$$

■

Note that the theorem above determined $\widehat{\Lambda}$ matrices completely, and hence the entries of S matrix as in definition 3.7[1]. By using Verlinde's formula, one can write down a series of equations of fusion rules in terms of $\widehat{\Lambda}$ matrices. Since fusion coefficients are non-negative integers, these equations describe certain arithmetic properties of $\widehat{\Lambda}$ matrices, and none of them seems to trivial for the case of conformal nets associated with $SU(n)$ at level k where S, T matrices are given (cf. [28]). We refer the reader to Cor. 9. 9 for such a statement in the case when $N = 2$.

4 Arithmetic properties of S, T matrices for a completely rational net

4.1 Galois action on $\widehat{\Lambda}$ matrices

In this section we'll study the Galois action in the cyclic permutation orbifold $\mathcal{D}^{\langle g \rangle}$ as in [1]. By Th. 2.7 the Galois action on the genus 0 S-matrix elements of $\mathcal{D}^{\langle g \rangle}$ may be described via suitable permutations π_l of the irreducible representations of the orbifold and signs ε_l. This will in turn allow us to determine the Galois action on $\widehat{\Lambda}$-matrices as defined in definition 3.7.

Let N be a positive integer , and as in §3 consider the cyclic permutation $g = (1, \ldots, N)$. We will use $C(N, \mathcal{A})$ to denote the conductor of the permutation orbifold $\mathcal{D}^{\langle g \rangle}$. Hence $C(1, \mathcal{A})$ is the conductor of $\mathcal{A}$. Note that $C(1, \mathcal{A})$ depends on the choice of $c_0(\mathcal{A})$ in definition 2.2, and our choice of $c_0(\mathcal{D}^{\langle g \rangle})$ is as in remark 2.13.

Among the irreducible representations of the permutation orbifold $\mathcal{D}^{\langle g \rangle}$ there is a subset $\mathcal{J}$ of special relevance to us. The elements in $\mathcal{J}$ are labeled by triples (λ, g^n, k), where λ is an irreducible representation of $\mathcal{A}$, while n and k are integers mod N. The subset of those (λ, g^n, k) where n is coprime to N will be denoted by $\mathcal{J}_0$. It follows from (1) of Lemma 2.10 that $(\lambda, g^n, k) \in \mathcal{J}_0$ have vanishing S-matrix elements with the labels not in $\mathcal{J}$, while for $(\mu, g^m, l) \in \mathcal{J}$ we have:

$$S_{(\lambda, g^n, k), (\mu, g^m, l)} = \frac{1}{N} \zeta_N^{-(km+ln)} \widehat{\Lambda}_{\lambda,\mu} \left(\frac{m\widehat{n}}{N} \right) \tag{12}$$

where $\widehat{n}$ denotes the mod N inverse of n and $\zeta_N = \exp\left(\frac{2\pi i}{N}\right)$.

Lemma 4.1. $\varepsilon_l(\tau_N(\lambda)) = \varepsilon_l(\lambda), \tau_N \pi_l(\lambda) = \pi_l(\tau_N \lambda)$.

Proof Note that by the property of τ_N we have $\frac{S_{\tau_N,\mu}}{S_{1,\mu}} = \pm 1$. By definition of Galois actions we have

$$\sigma_l(S_{\tau_N \lambda,\mu}) = \frac{S_{\tau_N,\mu}}{S_{1,\mu}} \sigma_l(S_{\lambda,\mu}) = \frac{S_{\tau_N,\mu}}{S_{1,\mu}} \varepsilon_l(\lambda) S_{\pi_l(\lambda),\mu} = \varepsilon_l(\tau_N \lambda) S_{\pi_l(\tau_N \lambda),\mu}$$

[1] With little effort we can in fact determine all entries of S matrix for the cyclic permutation orbifold using the methods of this chapter. However Th. 3.12 is enough for the purpose of this paper.

Hence

$$\varepsilon_l(\lambda)S_{\tau_N\pi_l(\lambda),\mu} = \varepsilon_l(\tau_N(\lambda))S_{\pi_l(\tau_N\lambda),\mu}$$

By unitarity of S matrix the lemma is proved. ■

By using Lemma 4.1 and Th. 3.12, the proofs of Lemma 1-3 Prop.1, Cor. 1 and Th. 1 of [1] go through (In the statements of Lemma 1-3, Prop. 1, Bantay's Λ matrix has to be replaced by our $\widehat{\Lambda}$ matrix, and the additional assumption on l is that l is coprime to $C(N,\mathcal{A})$ where N is the denominator of a rational number r as given in these statements)

For reader's convenience and to set up notations, we summarize Lemmas 1-3 and Prop. 1 of [1] in the following and sketch its proof.

Lemma 4.2. *Assume that l is coprime to the denominator N of r and $C(N,\mathcal{A})C(1,\mathcal{A})$. Then:*

(1) The set $\mathcal{J}$ is invariant under the permutations $\tilde{\pi}_l$, i.e. $\tilde{\pi}_l(\mathcal{J}) = \mathcal{J}$. For $(\lambda, g^n, k) \in \mathcal{J}_0$ one has

$$\tilde{\pi}_l(\lambda, g^n, k) = \left(\pi_l(\lambda), g^{ln}, \tilde{k}\right) \tag{13}$$

for some function $\tilde{k}$ of l, λ, n and k, and

$$\tilde{\varepsilon}_l(\lambda, g^n, k) = \varepsilon_l(\lambda) \tag{14}$$

(2)

$$\sigma_l\left(\widehat{\Lambda}(r)\right) = \widehat{\Lambda}(lr)\, G_l Z_l(r^*) = Z_l(r)\, G_l^{-1}\widehat{\Lambda}(\widehat{l}r) \tag{15}$$

where $Z_l(r)$ is a diagonal matrix whose order divides the denominator N of r, and $\widehat{l}$ is the mod N inverse of l, and $Z_l(0) = \mathbb{I}$, $Z_l(r+1) = Z_l(r)$;

(3)

$$G_l^{-1} Z_m(\widehat{l}r) G_l = Z_{lm}(r) Z_l^{-m}(r) \tag{16}$$

whenever both l and m are coprime to the denominator N of r and $C(N,\mathcal{A})C(1,\mathcal{A})$;

(4) If n is coprime to the denominator of r, then

$$Z_l^n(r) = Z_l(nr) \tag{17}$$

Proof We give a proof of (1) following the proof of Bantay indicating necessary changes. First, let's fix $(\lambda, g^n, k) \in \mathcal{J}$. According to Eq.(12), we have

$$S_{(\lambda,g^n,k),(\mu,1)} = \frac{1}{N}\zeta_N^{-k}\widehat{\Lambda}_{\lambda,\mu}\left(\frac{\hat{n}}{N}\right)$$

and this expression differs from 0 for at least one μ, by the unitarity of Λ-matrices and Th. 3.12. Select such a μ, and apply σ_l to both sides of the equation. One gets that

$$\tilde{\varepsilon}_l(\lambda, g^n, k)\, S_{\tilde{\pi}_l(\lambda,g^n,k),(\mu,1)} = \sigma_l\left(S_{(\lambda,g^n,k),(\mu,1)}\right)$$

differs from 0, but this can only happen if $\tilde{\pi}_l\,(\lambda, g^n, k] \in \mathcal{J}$ because $(\mu, 1) \in \mathcal{J}_0$.

Next, for $[\lambda, g^n, k] \in \mathcal{J}_0$ consider

$$S_{(\lambda,g^n,k),(\mu,1,m)} = \frac{1}{N}\zeta_N^{-nm} S_{\tau_N\lambda,\mu} \tag{18}$$

Applying σ_l to both sides of the above equation we get from Eq.(4)

$$\tilde{\varepsilon}_l(\lambda, g^n, k) S_{\tilde{\pi}_l(\lambda,g^n,k),(\mu,1,m)} = \frac{1}{N}\zeta_N^{-lnm}\varepsilon_l(\tau_N\lambda) S_{\pi_l(\tau_N\lambda),\mu}$$

But the lhs. equals

$$\tilde{\varepsilon}_l(\lambda, g^n, k)\frac{1}{N}\zeta_N^{-\tilde{n}m} S_{\tau_N\tilde{\lambda},\mu}$$

according to Eq.(18) if $\tilde{\pi}_l\,(\lambda, g^n, k) = \left[\tilde{\lambda}, g^{\tilde{n}}, \tilde{k}\right]$. Equating both sides we arrive at

$$S_{\tau_N\tilde{\lambda},\mu} = \varepsilon_l(\tau_N\lambda)\tilde{\varepsilon}_l(\lambda, g^n, k)\zeta_N^{-m(\tilde{n}-ln)} S_{\pi_l(\tau_N\lambda),\mu}$$

By Lemma 4.1 and the fact that

$$S_{\tau_N\lambda,\mu} = \frac{S_{\tau_N,\mu}}{S_{1,\mu}} S_{\lambda,\mu}$$

we have

$$S_{\tilde{\lambda},\mu} = \varepsilon_l(\lambda)\tilde{\varepsilon}_l(\lambda, n, k)\zeta_N^{-m(\tilde{n}-ln)} S_{\pi_l(\lambda),\mu}$$

As the lhs. is independent of m, we must have

$$\tilde{n} = ln \bmod N$$

and

$$\tilde{\lambda} = \pi_l(\lambda)$$

as well as

$$\tilde{\varepsilon}_l(\lambda, g^n, k) = \varepsilon_l(\lambda)$$

The proof of (2)-(4) is the same as that of Bantay with his Λ matrices replaced with our $\widehat{\Lambda}$ matrices. ■

Let us sketch the proof of the following (cf. Th. 1 in [1]) theorem, indicating modifications compared to the proof in [1]:

Theorem 4.3. *Let $\mathcal{A}$ be a completely rational net and let T-matrix be defined as after definition 2.2. Then for all l coprime to the conductor $G_l^{-1} T G_l = T^{l^2}$.*

Proof Let N be the order of T. Then N divides the conductor by definition. Choose l so that $(l, 12C(1,\mathcal{A})C(N,\mathcal{A}))=1$. By Th. 3.12 we have $\widehat{\Lambda}(\frac{1}{N})=\exp(\frac{-2\pi i(c-c_0)(N^2-1)}{12N})\Lambda(\frac{1}{N})$. Follow the argument of Bantay, with Λ replaced by $\widehat{\Lambda}$ we have

$$\exp(\frac{-2\pi i(c-c_0)(N^2-1)(l^2-1)}{12N})T^{\frac{-2l^2}{N}}=G_l^{-1}T^{\frac{-2}{N}}G_lZ_l^2(\frac{l}{N})$$

Now we use the fact that since $(l,12)=1, 12|l^2-1$. The rest of the argument is the same as [1] and we have $G_l^{-1}TG_l=T^{l^2}$ for l with the property $(l, 12C(1,\mathcal{A})C(N,\mathcal{A}))=1$. Now for any l coprime to the conductor $C(1,\mathcal{A})$, by Dirichelet theorem on arithmetic progressions we can always find integer p so that $l_1=l+pC(1,\mathcal{A})$ with the property that $(l, 12C(1,\mathcal{A})C(N,\mathcal{A}))=1$. Since $G_{l_1}=G_l, T^{l_1^2}=T^{l^2}$, the theorem is proved for any l coprime to the conductor. ■

This above theorem has been conjectured in [7], where some of its consequences had been derived.

Prop. 2 of [1] has to be modified due to phase factor as follows:

Proposition 4.4. *Let $r=\frac{n}{N}$. If l is coprime to $C(N,\mathcal{A})C(1,\mathcal{A})N$, then*

$$G_l^{-1}T^rG_l=T^{l^2r}Z_l^l(r)\exp(\frac{-2\pi i(l^2-1)(c-c_0)r}{24})$$

Proof The proof is similar to the proof of Prop. 2 in [1] and we indicate modifications when necessary. Write $r=\frac{n}{N}$. The idea is to apply Th. 4.3 to $\mathcal{D}^{\langle g\rangle}$ with the order of g equal to N. The phase factor comes in when we note that

$$T_{(\lambda,g^n,k)}=\zeta_N^{nk}T_\lambda^{\frac{1}{N}}\exp(\frac{2\pi i(c-c_0)}{24}(N-\frac{1}{N}))$$

Use Th. 4.3 we have

$$T_\lambda^{\frac{l^2}{N}}\exp(\frac{2\pi i(c-c_0)(l^2-1)}{24}(N-\frac{1}{N}))=\zeta_N^{lk_0}T_{\pi_l(\lambda)}^{\frac{1}{N}}$$

and the rest of the proof is as in [1] . ■

By using Th. 4.3, Prop. 3-6 and Cor. 2 of [1] follows in our setting with the same proof (except (4) in the theorem below)as in [1] and [7]. We record these results in the following theorem:

Theorem 4.5. *Let $\mathcal{A}$ be completely rational net and let S,T be its genus 0 modular matrices as defined after definition 2.2. Then:*

(1) For l coprime to the conductor,

$$G_l=S^{-1}T^lST^{\widehat{l}}ST^l \tag{19}$$

where $\widehat{l}$ denotes the inverse of l modulo the conductor;

(2) The conductor equals the order N of T, and $F = \mathbb{Q}[\zeta_N]$;

(3) Let N_0 denote the order of the matrix $\omega_0^{-1}T$, i.e. the least common multiple of the denominators of the conformal weights. Then $N = eN_0$, where the integer e divides 12. Moreover, the greatest common divisor of e and N_0 is either 1 or 2;

(4) N_0 times the central charge c is an even integer;

(5) There exists a function $N(r)$ such that the conductor N divides $N(r)$ if the number of irreducible representations of $\mathcal{A}$ - i.e. the dimension of the modular representation - is r.

Proof Given Th. 4.3, (1),(2), (3), and (5) are proved in the same way as in [1]. As for (4), the proof of Cor. 2 in [1] shows that N_0c_0 is an even integer. By Lemma 9.7 of [18] $c - c_0 \in 4\mathbb{Z}$ and (4) is proved. ■

4.2 The kernel of the modular representation

In this section we consider the modular representation of a completely rational net $\mathcal{A}$ as defined after Lemma 2.3. We will show that this representation factorizes through a congruence subgroup. We refer the reader to [7] for a nice account of this and related questions. Recall that the kernel $\mathcal{K}$ consists of those modular transformations which are represented by the identity matrix, i.e.

$$\mathcal{K} = \{m \in \Gamma(1) \,|\, M_{\lambda,\mu} = \delta_{\lambda,\mu}\}$$

Proposition 4.6. *If $m = \begin{pmatrix} a & b \\ e & d \end{pmatrix} \in \Gamma(1)$ with d coprime to the conductor [2]. Let m_e be an integer such that $m_e g(\frac{a}{e}) \in \mathbb{Z}$. Let $l = d + meC(1, \mathcal{A})$ be such that l is coprime to $6m_eC(|e|, \mathcal{A})$. Then*

$$\sigma_d(M) = T^bS^{-1}T^{-e}\sigma_d(S)\exp(-2\pi i(lg(\frac{a}{e}) - g(\frac{1}{e}) - \frac{(l^2-1)(c-c_0)}{24e}))$$

Proof According to Eq.(15) and Th. 3.12

$$\sigma_l(M) = \sigma_l(T^{a/e}\Lambda(\frac{a}{e})T^{d/e}) = \sigma_l(T^{a/e}\exp(-2\pi ig(\frac{a}{e}))\widehat{\Lambda}(\frac{a}{e})T^{d/e})$$

By our assumption on l we have

$$\sigma_l(T^{a/e}\exp(-2\pi ig(\frac{a}{e}))\widehat{\Lambda}(\frac{a}{e})T^{d/e}) = \exp(-2\pi i(lg(\frac{a}{e}) - g(\frac{1}{e})))\times \\ T^{ad/e}\Lambda(\frac{1}{e})G_lZ_l(\frac{d}{e})T^{d^2/e}$$

But

$$\Lambda(\frac{1}{e}) = T^{-1/e}S^{-1}T^{-c}ST^{-1/e}$$

[2]We use e instead of more natural c since c has been used to denote the central charge.

so

$$\sigma_l(M) = \exp(-2\pi i(lg(\frac{a}{e}) - g(\frac{1}{e})))\times$$
$$T^b S^{-1} T^{-e} S T^{-1/e} G_l Z_l(d/e) T^{\frac{d^2}{e}}$$

From Prop. 4.4

$$T^{-1/e} G_l = G_l T^{-l^2/e} Z_l^l(-1/e)\exp(\frac{2\pi i(l^2-1)(c-c_0)}{24e})$$

Putting all this together and using Lemma 4.2 we get the proposition. ∎

Next we show that the phase factor in the above proposition is always 1:

Proposition 4.7. *Let l be as in Prop. 4.6. Then*

$$\exp(-2\pi i(lg(\frac{a}{e}) - g(\frac{1}{e}) - \frac{(l^2-1)(c-c_0)}{24e})) = 1$$

Proof Let $4x = (c - c_0)$. By Lemma 9.7 of [18] x is an integer. Let us first prove the proposition for the case $x = 2x_2$ is even. Choose an integer n so that $3n + x_2 > 0$ and consider a local net $\mathcal{E}$ which is $3n + x_2$ tensor product of the local net $\mathcal{A}_{(E_8)_1}$. This net has μ index equal to one by Th. 3.18 of [9]. We choose our $c_0(\mathcal{E}) = 24n$ in the definition of T matrix for $\mathcal{E}$. The corresponding modular representation is trivial. We denote by $\widehat{\Lambda}_{\mathcal{E}}$ (resp. $\Lambda_{\mathcal{E}}$) the matrices as defined in definition 3.7 (resp. Bantay's Λ matrices) associated with $\mathcal{E}$. Apply Th. 3.12 to $\mathcal{E}$ we have

$$\widehat{\Lambda_{\mathcal{E}}}(a/e) = \exp(2\pi i g(a/e))\Lambda_{\mathcal{E}}(a/e)$$

By (2) of Th. 4.5 the conductor of the e-th cyclic permutation orbifold of $\mathcal{E}$ divides $3e$. By conditions on l we can apply Prop. 4.6 to the net $\mathcal{E}$ to have

$$\sigma_d(M) = T^b S^{-1} T^{-c}\sigma_d(S)\exp(-2\pi i(lg(\frac{a}{e}) - g(\frac{1}{e}) - \frac{(l^2-1)(c-c_0)}{24e}))$$

Since the modular representation for $\mathcal{E}$ is trivial we must have

$$\exp(-2\pi i(lg(\frac{a}{e}) - g(\frac{1}{e}) - \frac{(l^2-1)(c-c_0)}{24e})) = 1$$

and we have proved proposition for x even.

If $x = 2(x_1 + 2) - 3$ is odd, we define a new set of S_1, T_1 matrix by

$$T_1 = \exp\frac{-2\pi i x}{6}T, S_1 = \exp\frac{-2\pi i x}{2}T$$

and denote by Λ_1 the Λ matrix of Bantay associated with S_1, T_1. Let $g_1(k/n)$ be defined modulo integers such that $\Lambda_1 = \exp(2\pi i g_1(k/n))\Lambda$. From the definition 3.11 one checks easily that modulo integers

$$g(k/n) = \frac{n}{2} + g_1(k/n)$$

Using the assumption that l is odd it is now sufficient to check the proposition for g_1. From the defining equation for g_1, we see that $\exp 2\pi i g_1(k/n)$ is Bantay's Λ matrix associated with one dimensional representation of the modular group given by $T \to \exp\frac{-2\pi i x}{6}, S \to \exp\frac{-2\pi i x}{2}$. This representation is the tensor product of two one dimensional representations given by

$$T \to \exp(\frac{-2\pi i(2x_1+4)}{6}), S \to 1$$

and

$$T \to \exp\frac{2\pi i}{2}, S \to \exp\frac{2\pi i}{2}$$

and we denote by g_3, g_2 the associated Λ matrices. Note that $g_1(a/e) = g_2(a/e) + g_3(a/e)$. The same proof as in the x even case, with $x_2 = x_1 + 2$, shows that

$$\exp(-2\pi i(lg_3(\frac{a}{e}) - g_3(\frac{1}{e}) - \frac{(l^2-1)(2x_1+4)}{24e})) = 1$$

Hence to finish the proof we just have to show that

$$\exp(-2\pi i(lg_2(\frac{a}{e}) - g_2(\frac{1}{e}) + \frac{(l^2-1)}{2e})) = 1$$

Since the associated modular representation is very simple, this can be checked directly using the following formulas:

$$\exp(2\pi i g_2(a/e)) = \exp(\frac{-2\pi i(a+d)}{2e})M_2(a,b,e,d)$$

where $M_2(a,b,e,d) = \pm 1$. When e or d is even, $M_2(a,b,e,d) = (-1)^d$ or $(-1)^e$; When e, d are odd, $M_2(a,b,e,d) = (-1)^{a+d+1}$. ■

By combining the above two propositions we have proved the following:

Theorem 4.8. *If d is coprime to the conductor, then*

$$\sigma_d(M) = T^b S^{-1} T^{-e} \sigma_d(S).$$

Now Th. 2-4 of [1] follow exactly in the same way. Let us record these theorems in the following:

Theorem 4.9. *Let $\mathcal{A}$ be a completely rational net and consider the modular representation as defined after Lemma 2.3. Then:*

(1) Let d be coprime to the conductor N. Then $\begin{pmatrix} a & b \\ e & d \end{pmatrix} \in \Gamma(1)$ belongs to the kernel $\mathcal{K}$ if and only if

$$\sigma_d(S)\, T^b = T^e S; \tag{20}$$

(2) Define

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ e & d \end{pmatrix} \in \Gamma(1) \mid a, d \equiv 1 mod\ N,\ e \equiv 0 mod\ N \right\}$$

and

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ e & d \end{pmatrix} \in \Gamma_1(N) \mid b \equiv 0 mod N \right\}$$

Then

$$\mathcal{K} \cap \Gamma_1(N) = \Gamma(N)$$

In particular, $\mathcal{K}$ is a congruence subgroup of level N;

(3) Define $SL_2(N) \cong \Gamma(1)/\Gamma(N)$. The modular representation factorizes through $SL_2(N)$ which we denote by D. For l coprime to N, define the automorphism τ_l : $SL_2(N) \to SL_2(N)$ by

$$\tau_l \begin{pmatrix} a & b \\ e & d \end{pmatrix} = \begin{pmatrix} a & lb \\ \widehat{l}e & d \end{pmatrix} \tag{21}$$

where $\widehat{l}$ is the mod N inverse of l. Then $\sigma_l \circ D = D \circ \tau_l$.

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